

§ Covariant derivatives

Given a surface $S \subseteq \mathbb{R}^3$, recall that a vector field on S

$$X : S \longrightarrow \mathbb{R}^3 \quad (\text{smooth})$$

- is **tangential** if $X_p \in T_p S \quad \forall p \in S$
- is **normal** if $X_p \in (T_p S)^\perp \quad \forall p \in S$

Defⁿ: $\mathfrak{X}(S) := \{ \text{tangential vector fields on } S \}$

$$\mathfrak{X}^\perp(S) := \{ \text{normal vector fields on } S \}$$

Q: How to differentiate vector fields in $\mathfrak{X}(S)$?

A: Covariant derivatives !

Defⁿ: Given $X, Y \in \mathfrak{X}(S)$, define the **covariant derivative of Y along X** as

$$\nabla_X Y := (D_X Y)^T$$

where $(\cdot)^T$ refers to the tangential component of a vector based at $p \in S$ according to the orthogonal splitting

(depends
on P)

$$(\mathbb{R}^3 =) T_P \mathbb{R}^3 = T_P S \oplus (T_P S)^\perp \quad (*)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

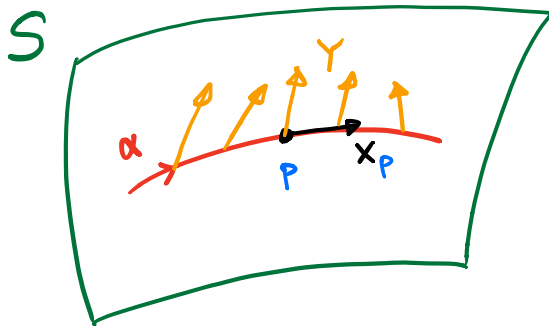
$$V = V^T + V^\perp$$

Remarks: (1) Recall that $D_X Y(P)$ depends ONLY on

(a) the vector X_P

and (b) the values of Y restricted to ANY curve

$$\alpha: (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}^3 \text{ s.t. } \alpha(0) = P, \alpha'(0) = X_P$$



$$D_X Y(P) = \left. \frac{d}{dt} Y(\alpha(t)) \right|_{t=0}$$

Hence, this is well-defined even

X, Y are only defined on S .

$$(2) \quad X, Y \in \mathfrak{X}(S) \Rightarrow \nabla_X Y \in \mathfrak{X}(S)$$

We now study some important properties of ∇ .

Properties of ∇ : Let $X, Y, Z \in \mathfrak{X}(S)$, $f \in C^\infty(S)$,
 a, b are real constants.

(1) Linearity in both variables:

$$\nabla_X (aY + bZ) = a \nabla_X Y + b \nabla_X Z$$

$$\nabla_{aX + bY} Z = a \nabla_X Z + b \nabla_Y Z$$

(2) Leibniz rule: $\nabla_X (fY) = X(f)Y + f \nabla_X Y$

(3) Tensorial: $\nabla_{fX} Y = f \nabla_X Y$

(4) Torsion free: $\nabla_X Y - \nabla_Y X = [X, Y]$

(5) Metric compatibility:

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle X, \nabla_X Z \rangle$$

Remark: The covariant derivative

$$\nabla : \mathfrak{X}(S) \times \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)$$

$$X, Y \longmapsto \nabla_X Y$$

is uniquely defined by properties (1) – (5) above!

"Fundamental Theorem of Riemannian geometry"

Proof: It follows from the fact that (1) - (5) are satisfied with " ∇ " replaced by "D" for vector fields in $\mathbb{R}^3$.

E.g. To prove (5),

$$X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle$$

$$\text{(Since } Y, Z \in \mathfrak{X}(S) \text{)} = \underbrace{\langle (D_X Y)^T, Z \rangle}_{\substack{= \\ \nabla_X Y}} + \underbrace{\langle Y, (D_X Z)^T \rangle}_{\substack{= \\ \nabla_X Z}}$$

_____ $\square$

§ Einstein summation convention

- 2 kinds of indices: **upper** & **lower**
- Same index appearing BOTH as an upper & lower index in the same term $\implies$ Sum over this index

E.g: coordinates in $\mathbb{R}^n$: $x^1, x^2, \dots, x^n$ (upper)

Coordinate vector fields in $\mathbb{R}^n$: $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ (lower)

vector fields in $\mathbb{R}^n$: $X = \sum_{i=1}^n a^i \frac{\partial}{\partial x^i} = a^i \frac{\partial}{\partial x^i}$
↑
Einstein
summation
convention

E.g. $\Sigma(u^1, u^2) : \mathcal{U} \rightarrow S \subseteq \mathbb{R}^3$ parametrization

write

$$\partial_i := \frac{\partial \Sigma}{\partial u^i}$$

Coordinate vector fields

Any **tangential** vector field $X \in \mathfrak{X}(S)$ can be locally

expressed as

$$X = a^i \partial_i = a^k \partial_k$$

"dummy index"

1st f.f. : $g_{ij} = \langle \partial_i, \partial_j \rangle$

inverse: $(g^{ij}) = (g_{ij})^{-1}$

2nd f.f. : $A_{ij} = \langle \frac{\partial^2 \Sigma}{\partial u^i \partial u^j}, N \rangle$

E.g. $\langle a^i \partial_i, b^j \partial_j \rangle = g_{ij} a^i b^j$

$$g^{ik} g_{kj} = \delta_{ij}$$

§ Christoffel Symbols

Given a parametrization $\mathbf{x}(u^1, u^2)$ on S

tangent vectors: $\partial_1 = \frac{\partial \mathbf{x}}{\partial u^1}$, $\partial_2 = \frac{\partial \mathbf{x}}{\partial u^2}$

unit normal: $\mathbf{N} = \frac{\partial_1 \times \partial_2}{\|\partial_1 \times \partial_2\|}$

At each point $p \in S$, we have the orthogonal splitting:

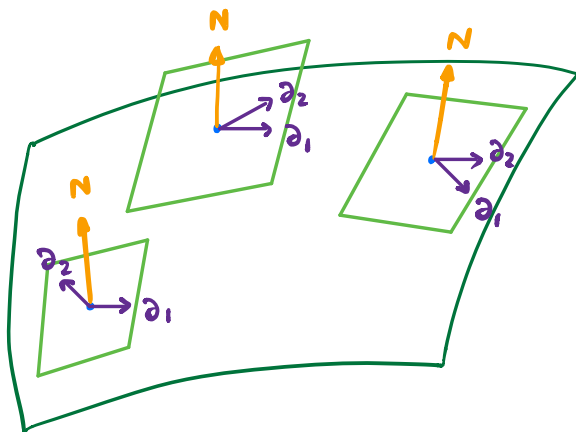
$$\mathbb{R}^3 \cong T_p \mathbb{R}^3 = \underbrace{T_p S}_{\partial_1, \partial_2} \oplus \underbrace{(T_p S)^\perp}_{\mathbf{N}}$$

basis: $\partial_1, \partial_2 \perp \mathbf{N}$

Caution: NOT orthonormal!

$$(g_{ij}) = \langle \partial_i, \partial_j \rangle \neq \delta_{ij}$$

We have a 2-parameter family of basis of $\mathbb{R}^3$ along S :



Similar to a "moving frame", but NOT orthonormal!

Defⁿ: The Christoffel symbols Γ_{ij}^k of a given coordinate system u^1, u^2 is defined as

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k$$

Prop: (1) $\Gamma_{ij}^k = \Gamma_{ji}^k$ (symmetry)

(2) $\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij})$ Koszul formula

Proof: (1) Since ∇ is torsion-free, i.e.

$$\underbrace{\nabla_{\partial_i} \partial_j - \nabla_{\partial_j} \partial_i}_{=0} = [\partial_i, \partial_j] = 0$$

$$(\underbrace{\Gamma_{ij}^k - \Gamma_{ji}^k}_{=0}) \partial_k = 0$$

(2) Since ∇ is metric compatible,

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

for any tangential

vector fields $X, Y, Z \in \mathfrak{X}(S)$

In particular, if we choose $X = \partial$, $Y = \partial_i$, $Z = \partial_j$

$$\begin{aligned}\Rightarrow \partial_l g_{ij} &= \langle T_{li}^k \partial_k, \partial_j \rangle + \langle \partial_i, T_{lj}^k \partial_k \rangle \\ &= T_{li}^k g_{kj} + T_{lj}^k g_{ik}\end{aligned}$$

Cyclicly permuting i, j, l , we obtain using (1)

$$\begin{aligned}- \partial_l g_{ij} &= \cancel{T_{li}^k g_{kj}} + \cancel{T_{lj}^k g_{ik}} \\ + \partial_i g_{jl} &= T_{ij}^k g_{kl} + \cancel{T_{il}^k g_{jk}} \\ + \partial_j g_{li} &= \cancel{T_{jl}^k g_{ki}} + T_{ji}^k g_{lk}\end{aligned}$$

$$\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij} = 2 g_{kl} T_{ij}^k$$

Multiplying by g^{ml} on both sides

$$g^{ml} (\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ij}) = 2 \underbrace{g^{ml} g_{kl}}_{\delta^m_k} T_{ij}^k = 2 T_{ij}^m$$

Dividing 2 on both sides and switch m to k gives the desired formula.

_____ $\square$

Remark: An important feature of **Koszul formula** is that we can compute T_{ij}^k directly from g_{ij} and its first derivatives, i.e.

$$T_{ij}^k = F(g_{ij}, \partial_k g_{ij})$$

Now, we can derive some sort of "Frenet-type" formulas by differentiating the moving basis $\{\partial_1, \partial_2, \mathbf{N}\}$.

Prop: (Gauss)

$$D_{\partial_i} \partial_j = T_{ij}^k \partial_k + A_{ij} \mathbf{N}$$

(Weingarten)

$$D_{\partial_i} \mathbf{N} = -g^{jk} A_{ij} \partial_k$$

(*)

Proof:

$$\begin{aligned} D_{\partial_i} \partial_j &= (D_{\partial_i} \partial_j)^T + (D_{\partial_i} \partial_j)^N \\ &= \nabla_{\partial_i} \partial_j + \langle D_{\partial_i} \partial_j, \mathbf{N} \rangle \mathbf{N} \\ &= T_{ij}^k \partial_k + A_{ij} \mathbf{N} \end{aligned}$$

This proves the first equation. For the second equation,

first notice that $\langle \mathbf{N}, \mathbf{N} \rangle \equiv 1$, therefore

$$0 = \partial_i \langle \mathbf{N}, \mathbf{N} \rangle = 2 \langle D_{\partial_i} \mathbf{N}, \mathbf{N} \rangle \quad \text{i.e. } D_{\partial_i} \mathbf{N} \text{ is tangential!}$$

On the other hand, since $\langle \mathbf{N}, \partial_j \rangle \equiv 0$,

$$\langle D_{\partial_i} \mathbf{N}, \partial_j \rangle = - \langle \mathbf{N}, D_{\partial_i} \partial_j \rangle = - A_{ij}$$

So, if we let $D_{\partial_i} \mathbf{N} = C_i^k \partial_k$, then

$$C_i^k g_{kj} = - A_{ij}$$

$$\Rightarrow C_i^k = - g^{kj} A_{ij}$$

This proves the second equation.

_____ $\square$

Remark: From (*) we see that the second f.f A_{ij} can be obtained from the "rate of change of the moving basis $\{\partial_1, \partial_2, \mathbf{N}\}$ ", which in turn gives the Gauss and mean curvatures K and H .

§ Gauss and Codazzi equations

We are now ready to prove one of the most important theorems in classical differential geometry.

Theorema Egregium: (Gauss' Golden Theorem)

The Gauss curvature K is an intrinsic invariant, i.e. it depends only on the 1st f.f. g_{ij} (and its higher derivatives)

The theorem above follows from a set of equations called "constraint equations". To understand these equations, we look at the following:

Problem: Given an open set $U \subseteq \mathbb{R}^2$, and two smooth family of matrices defined on U s.t.

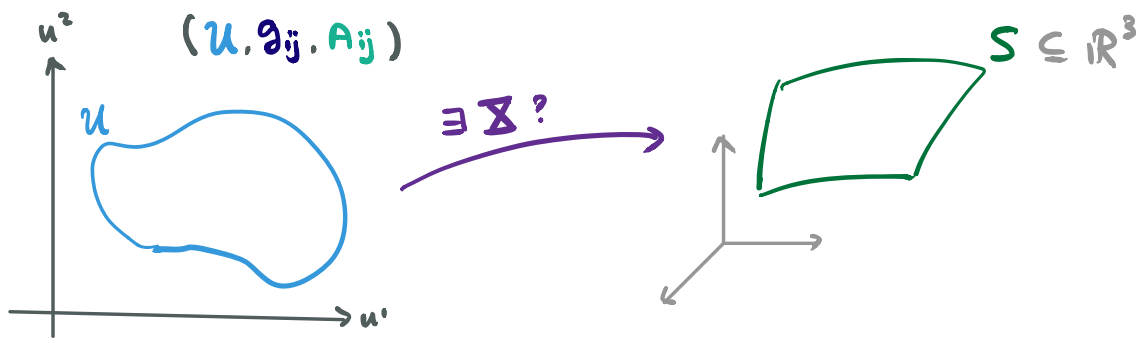
(g_{ij}) : symmetric & positive definite

(A_{ij}) : symmetric

Can one find a parametrization $\Sigma: U \rightarrow S \subseteq \mathbb{R}^3$ of a surface S s.t.

$(g_{ij}) = 1^{\text{st}}$ f.f. and $(A_{ij}) = 2^{\text{nd}}$ f.f.

?



Recall: For curves in $\mathbb{R}^2$ or $\mathbb{R}^3$, one can prescribe any curvature and torsion.

The situation is drastically different that the f.f.'s have to satisfy a natural set of "compatibility equations".

Theorem: If (g_{ij}) and (A_{ij}) are the 1st & 2nd f.f. of a surface in $\mathbb{R}^3$ under some local coordinate system, then we have:

Gauss equation:

$$\partial_k T_{ij}^l - \partial_j T_{ik}^l + T_{ij}^p T_{pk}^l - T_{ik}^p T_{pj}^l = g^{lp} (A_{ij} A_{kp} - A_{ik} A_{jp})$$

Codazzi equation:

$$\partial_k A_{ij} - \partial_j A_{ik} + T_{ij}^p A_{pk} - T_{ik}^p A_{pj} = 0$$

Proof: Let $\Sigma(u^1, u^2) : \mathcal{U} \rightarrow \mathbb{R}^3$ be a parametrization.

Recall the **Gauss** & **Weingarten** equations:

$$\begin{cases} \partial_i \partial_j \Sigma = T_{ij}^k \partial_k \Sigma + A_{ij} N \\ \partial_i N = -g^{jk} A_{ij} \partial_k \Sigma \end{cases}$$

Since partial derivatives of any order commute:

$$\partial_k (\partial_i \partial_j \Sigma) = \partial_j (\partial_i \partial_k \Sigma) \quad (\#)$$

We will show that **Gauss** & **Codazzi** equations follow from equating the **tangential** and **normal** components respectively in $(\#)$.

$$\partial_k (\partial_i \partial_j \Sigma) \stackrel{\text{Gauss}}{=} \partial_k (T_{ij}^l \partial_l \Sigma + A_{ij} N)$$

$$\begin{aligned} &= (\partial_k T_{ij}^l) \partial_l \Sigma + T_{ij}^l (\partial_k \partial_l \Sigma) \\ &\quad + (\partial_k A_{ij}) N + A_{ij} (\partial_k N) \end{aligned}$$

$$\begin{aligned} &\stackrel{\text{Gauss}}{=} (\partial_k T_{ij}^l) \partial_l \Sigma + T_{ij}^l (T_{kl}^p \partial_p \Sigma + A_{kl} N) \\ &\stackrel{\text{Weingarten}}{=} (\partial_k T_{ij}^l) \partial_l \Sigma + T_{ij}^l (T_{kl}^p \partial_p \Sigma + A_{kl} N) \\ &\quad + (\partial_k A_{ij}) N + A_{ij} (-g^{pl} A_{kp} \partial_l \Sigma) \end{aligned}$$

(grouping tangential and normal terms
and renaming some dummy indices)

$$\partial_k(\partial_i \partial_j \mathbf{x}) = (\partial_k T_{ij}^l + T_{ij}^p T_{kp}^l - g^{pl} A_{ij} A_{kp}) \partial_l \\ + (\partial_k A_{ij} + T_{ij}^l A_{kl}) \mathbf{N}$$

Switching the indices k & j , we obtain

$$\partial_j(\partial_i \partial_k \mathbf{x}) = (\partial_j T_{ik}^l + T_{ik}^p T_{jp}^l - g^{pl} A_{ik} A_{jp}) \partial_l \\ + (\partial_j A_{ik} + T_{ik}^l A_{jl}) \mathbf{N}$$

Equating the tangential part $\Rightarrow$ Gauss equation.

Equating the normal part $\Rightarrow$ Codazzi equation.

□

It turns out that these are all the equations constraining (g_{ij}) and (A_{ij}) .

Bonnet's Theorem: Given (g_{ij}) & (A_{ij}) on an open set $\mathcal{U}$ of $\mathbb{R}^2$ satisfying Gauss & Codazzi equations, then

$$\exists \text{ parametrization } \mathbf{x}: \mathcal{U} \rightarrow \mathbb{R}^3$$

with $(g_{ij}) = 1^{\text{st}}$ f.f. & $(A_{ij}) = 2^{\text{nd}}$ f.f.

Proof: Omitted.

Proof of Theorema Egregium: (i.e. K is intrinsic.)

Recall:
$$K = \frac{\det(A_{ij})}{\det(g_{ij})}$$

It suffices to show that $\det(A_{ij}) = A_{11}A_{22} - A_{12}^2$ is intrinsic (i.e. depends only on g_{ij} and its derivatives)

Take $i=j=1$, $k=l=2$ in Gauss equation:

$$\begin{aligned}\partial_2 T_{11}^2 - \partial_1 T_{12}^2 + T_{11}^p T_{p2}^2 - T_{12}^p T_{p1}^2 &= g^{2p} (A_{11} A_{2p} - A_{12} A_{1p}) \\ &= g^{21} (\underbrace{A_{11} A_{21} - A_{12} A_{11}}_{=0}) + g^{22} (A_{11} A_{22} - A_{12} A_{12})\end{aligned}$$

$$\Rightarrow \det(A_{ij}) = \frac{1}{g^{22}} \left(\underbrace{\partial_2 T_{11}^2 - \partial_1 T_{12}^2 + T_{11}^p T_{p2}^2 - T_{12}^p T_{p1}^2}_{\text{depends only on } g_{ij} \text{ (up to 2nd derivatives)}} \right)$$

_____ $\square$